Lecture 18

Ordinary Differential Equations (ODEs)

A <u>differential equation</u> is an equation relating an unknown function to one or more of its derivatives. In the case that the unknown function has only one independent variable, it is called an ODE. These, in general, look like

$$F(x, y, y', y'', \cdots, y^{(n)}) = 0 \qquad (*)$$

By adding in initial values,

$$y(x_0) = y_0,\ y'(x_0) = y_1,\ \cdots,\ y^{(n-1)}(x_0) = y_{n-1} \qquad (**)$$

we get an <u>initial value problem</u>, or IVP.

The <u>order</u> of an ODE is the order of the highest derivative occuring in the equation. A <u>solution</u> of a differential equation $(*)$ is a function $f(x)$ such that $y = f(x)$ satisfies $(*)$, i.e.,

$$F(x, f(x), f'(x), \cdots, f^{(n)}(x)) = 0$$

$y = f(x)$ is the solution to the <u>initial value problem</u> if it satisfies each equation in $(**)$.

Ex: Here are some common ODEs. Give their order.

ⓐ Exponential growth: $y' = ky$, $y(0) = y_0$

Order: ___

ⓑ Kirchoff's second law: $\begin{cases} LQ'' + RQ' + \frac{1}{C}Q = E(t) \\ Q(t_0) = Q_0,\ Q'(t_0) = I_0 \end{cases}$
(Q is charge)

Order: ___

ⓒ (This one is made up): $y^{(4)} + 2x(y''')^3 - 2y''y' = 4e^x$

Order: ___

Ex: Determine whether the following functions are solutions of the ODE/IVP:

ⓐ $y' + y = 3$

ⓘ $y = e^{-x}$

ⓘⓘ $y = e^{-x} + 3$

ⓑ $\begin{cases} y' + \frac{y}{x} = 2e^{x^2} \\ y(1) = 1 \end{cases}$

ⓘ $y = \dfrac{e^{x^2}}{x}$

ⓘⓘ $y = \dfrac{e^{x^2} - e + 1}{x}$

9.2 - Direction Fields and Euler's Method

A direction (or slope) field for a first order differential equation is a collection of short line segments through points (x,y) in the plane with slope given by $y'(x,y)$. (Here, we solve our 1^{st} order ODE for y' to see it as a function of x & y, i.e., $y' = F(x,y)$.) The more points with lines through them that we have, the more accurate of a picture we get.

Ex: Sketch a slope field for $y' = 2x$.

If you notice, you can see curves traced out by these direction fields. These curves are the actual solutions to the ODE.

Now, it isn't always possible to solve an ODE, so the next best thing is to approximate solutions. Since we know that solutions follow the direction field, we can use this to linearly approximate the solutions. This method is called Euler's method.

Euler's Method

Suppose we have the IVP $\begin{cases} y' = F(x,y) \\ y(x_0) = y_0 \end{cases}$, and we want to know the value of its solution at some $x=a$ (typically $a > x_0$, but $a < x_0$ is fine too).

① Choose your step size, h. h should be chosen so that $a = x_0 + nh$ for some integer $n > 0$ (n is the number of steps that will be taken.) Generally, a smaller h leads to a better approximation.

② Starting at (x_0, y_0), we draw a line with slope $F(x_0, y_0)$ ending at $x_1 = x_0 + h$. The y-value at this point is $y_1 = y_0 + F(x_0, y_0)h$. y_1 is the approximation to the solution at x_0+h, i.e., an approximation of $y(x_0+h)$.

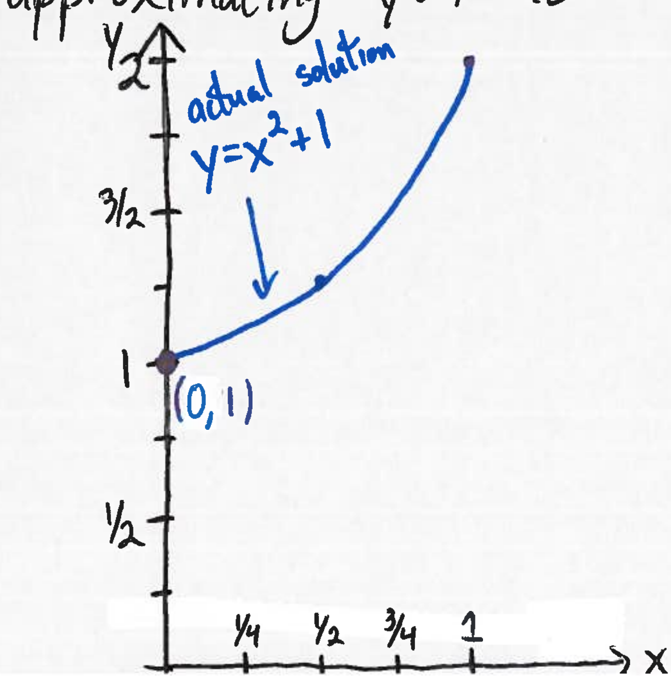
③ Repeat this process until we get to a:

$$x_2 = x_1 + h = x_0 + 2h, \quad y_2 = y_1 + F(x_1, y_1)h$$

$$\vdots \qquad\qquad \vdots$$

$$\boxed{x_n = x_{n-1} + h = x_0 + nh = a, \quad y_n = y_{n-1} + F(x_{n-1}, y_{n-1})h}$$

E.g., for $y' = 2x$, using an initial value $y(0) = 1$ and a step size $h = 0.25$, the picture behind approximating $y(1)$ is:



$n=$	$x_n = x_0 + nh$	$y_n = y_{n-1} + F(x_{n-1}, y_{n-1})h$
0		
1		
2		
3		
4		

Ex: Using Euler's method with a step size of $h=0.1$, estimate the solution to the IVP

$$\begin{cases} y' = x^2 + y \\ y(1) = 0 \end{cases}$$

at $x = 1.3$.